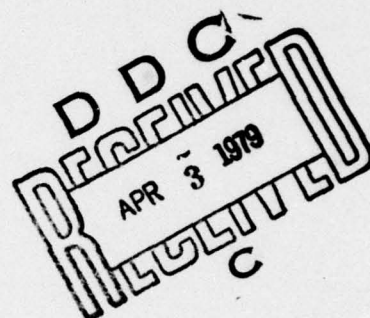SCHOOL OF OPERATIONS RESEARCH
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VALUE THEORY WITHOUT EFFICIENCY

by

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## Abstract

An axiomatic study of value theory without efficiency is presented.
A characterization of the class of operators that is obtained by omitting
from the value axioms the one about efficiency is given.  This is done
for the finite case as well as for some important spaces of non atomic games.

## 1. Introduction

Several recent researchers have treated generalizations or analogues of the Shapley value that do not enjoy the efficiency property (Banzhaf, Shapley-Dubey, Dubey, Roth). It is the purpose of this paper to treat the subject from an axiomatic viewpoint, i.e. to characterize the class of operators that is obtained by omitting the efficiency axiom from the axioms defining the Shapley value. We will treat both the finite and the non atomic case; in the finite case a complete solution will be given. In the non-atomic case, a complete solution is given for the important space pNA and we bring a characterization of all the continuous semi values on a class of political economic games.

## 2. The Finite Case

Let $U$ be an infinite set--the universe of players. A $\underline{game}$ is a set function $v: 2^U \to R$ with $v(\emptyset) = 0$. We shall refer to $v$ as a game on $U$. A set $T \subset U$ is a $\underline{support}$ of $v$ if for each $S \subset U$, $v(S) = v(S \cap T)$. A $\underline{finite\ game}$ is a game which has a finite support. The members of $U$ are called $\underline{players}$ and the members of $2^U$ $\underline{coalitions}$. The space of all finite games will be denoted by $G$. The subspace of $G$ of all additive games is denoted AG. A game is $\underline{monotonic}$ if $v(S \cup T) \geq v(S)$ for each $T, S \subset U$. A $\underline{permutation}$ of $U$ is a 1-1 function from $U$ onto itself; if $\theta$ is a permutation, $v \in G$ define the game $\theta^*v$ by $(\theta^*v)(S) = v(\theta S)$. Given $v$ in $G$ a $\underline{"dummy"\ player}$ of $v$ is a member $i$ of $U$ such that $v(S \cup \{i\}) = v(S) + v(\{i\})$ whenever $i \notin S$. A $\underline{semi\ value}$ on $G$ is then a function $\psi: G \to AG$ such that:

(2.1) $\psi$ is linear

(2.2) $\psi(\theta^*v) = \theta^*\psi v$, for each permutation $\theta$ of $U$.

(2.3) if $v$ is monotonic then $\psi v$ is monotonic.

(2.4) if $i$ is a "dummy" player of $v$

then $\psi v(\{i\}) = v(\{i\})$.

Conditions (2.1),(2.2),(2.3) and (2.4) are called the linearity, symmetry, positivity and dummy axioms respectively.

To state our characterization of the family of semi values on  G  it would be convenient to introduce an auxiliary probability space:  Let  $(\Omega,\mathcal{B},P)$  be a probability space, and  $X_i$  $(i \in U)$  is a family of independent identically distributed random variable distributed uniformly on [0,1].  If  $v \in G$  and  $t \in [0,1]$  define the random variable  $\Delta v(t)$  by:  $\Delta v(t) = v(\{i: X_i \leq t\}) - v(\{i: X_i < t\})$.  We are now ready to state our result.

Theorem 2.5:  For each probability measure  $\lambda$  on  [0,1]  there is a semi value  $\psi_\lambda$  on  G  defined by:

$$\psi_\lambda v(\{i\}) = \int_0^1 E(\Delta v(t) | X_i = t) \cdot d\lambda(t)$$

Moreover, any semi value on  G  is of that form and the mapping  $\lambda \rightarrow \psi_\lambda$  is 1-1.

To prove our theorem we first characterize the semi values on the vector space of games on a fixed finite player set.  Let  N  be a finite set.  The space of games on  N  is denoted by  $G^N$  (Note that  $G^N$  can be regarded as the subspace of  G  of all games  v  supported by  N.)  and similarly  $AG^N$.  A semi value on  $G^N$  is a function  $\psi: G^N \rightarrow AG^N$  satisfying (2.1),(2.2),(2.3),(2.4).  This characterization is done by means of the orders on  N.  By an order on  N,  we mean a total order, so that there are  n!  distinct orders on  N.  If  R  is such an order, and if  $i \in N$,  we denote by  $P_i^R$  the set of players preceding  i  in  R.  Then we have

Lemma 2.6:  For any function  $\gamma: \{0,\ldots,n-1\} \rightarrow R^+$  such that

(2.7)
$$\sum_{k=0}^{n-1} \gamma(k) = n$$

there exists a semi value  $\psi_\gamma$  on  $G^N$  defined by

$$(2.8) \qquad \psi_\gamma v(\{i\}) = \frac{1}{n!} \sum_R \gamma(|P_i^R|) \cdot [v(P_i^R \cup \{i\}) - v(P_i^R)]$$

Moreover, any semi value on $G^N$ is of that form and the mapping $\gamma \to \psi_\gamma$ is 1-1.

  <u>Proof</u>. It is easily verified that for each $\gamma: \{0,\ldots,n-1\} \to \mathbb{R}$ $\psi_\gamma$ is linear, symmetric, and for each "dummy" $i$ with $v(\{i\}) = 0$, $\psi_\gamma(\{i\}) = 0$. If $\gamma$ satisfies (2.6) then $\psi_\gamma$ obey the "dummy" axiom and if $\gamma: \{0,\ldots,n-1\} \to \mathbb{R}$ then $\psi_\gamma$ is monotonic. Now let $\psi$ be a semi value of $G^N$. For each $f: \{0,\ldots,n\} \to \mathbb{R}$ with $f(0) = 0$ define $v_f$ in $G^N$ by $v_f(S) = f(|S|)$. From (2.2) it follows that there exists $x(f)$ with $v_f(\{i\}) = x(f)$. Let $f_i: \{0,\ldots,n-1\} \to \mathbb{R}$ $i = 0,\ldots,n-1$ be defined by $f_i(k) = 1$ if $k > i$ and denote $\gamma(k) = x(f_k) \cdot n$. We shall show that for each $v$ in $G^N$, $\psi v = \psi_\gamma v$. First observe that $\psi v_{f_i} = \psi_\gamma v_{f_i}$ and as $v_{f_i}$ span the subspace of all games $v_f$ in $G^N$, (2.1) implies that $\psi v_f = \psi_\gamma v_f$ for each $f: \{0,\ldots,n\} \to \mathbb{R}$ with $f(0) = 0$.

  Define for each non-empty $T \subset N$ a game $u_T \in G^N$ by $u_T(S) = 1$ if $S \supset T$ and $u_T(S) = 0$ otherwise. From (2.2) it follows that $0 = \psi u_T(\{i\}) - \psi u_T(\{j\}) = \psi_\gamma u_T(\{i\}) - \psi_\gamma u_T(\{j\})$ whenever $i$ and $j$ are in $T$. From (2.4) it follows $\psi u_T(\{i\}) = \psi_\gamma u_T(\{i\}) = 0$ whenever $i \notin T$. Hence, to prove that $\psi u_T = \psi_\gamma u_T$ it is enough to show that $\psi u_T(N) = \psi_\gamma u_T(N)$, but if $T_1, T_2$ are coalitions with $|T_1| = |T_2|$ it follows that $\psi u_{T_1}(N) = \psi u_{T_2}(N)$ and $\psi_\gamma u_{T_1}(N) = \psi_\gamma u_{T_2}(N)$ and thus to prove that $\psi u_T = \psi_\gamma u_T$ it is enough to prove that $\psi u = \psi_\gamma u$ where $u = \sum_{\substack{S \subset N \\ |S| = |T|}} u_S$, but $u = v_f$ for some $f$, therefore we conclude that $\psi u_T = \psi_\gamma u_T$. As the games $u_T$ form a basis for the linear space $G^N$ (for a proof see [3] Appendix A), $\psi_\gamma = \psi$. Positivity implies that $\gamma(k) \geq 0$ and if $v \in G^N$ is defined by $v(S) = 1$ if $i \in S$ and otherwise $v(S) = 0$ then

$\psi_\gamma(i) = \frac{1}{n} \sum \gamma(h)$, thus (2.4) implies that $\gamma$ satisfies (2.6). To show that $\gamma \to \psi_\gamma$ is one to one, assume that $\gamma \neq \gamma'$ i.e., there exists $k$ with $\gamma(k) \neq \gamma(k')$ then a simple calculation shows that $\psi v_{f_k} \neq \psi v_{f_k}$.

Q.E.D.

Remark: In the bounded variation norm we have $\| \psi_\gamma \| = \max\limits_{0 < k \leq n-1} \gamma(k)$. We omit the simple verification. (That $\| \psi_\gamma \| \leq \max \gamma(k)$ follows from (2.8) while the reverse inequality can be obtained by considering the games $v_{f_k}$.)

The following is a reformulation of lemma 2.5:

Lemma 2.6*: For any function $c: \{0,\dots,n-1\} \to \mathbb{R}^+$ such that

$$(2.9) \qquad \sum_{k=0}^{n-1} c(k) \binom{n-1}{k} = 1$$

there exists a semi value $\psi_c$ on $G^N$ defined by

$$(2.10) \qquad \psi_c v(\{i\}) = \sum_{S \subset N \setminus \{i\}} c(|S|) \cdot [v(S \cup \{i\}) - v(S)].$$

Moreover, any semi value on $G^N$ is of that form and the mapping $c \to \psi_c$ is 1-1.

We omit the simple verification (set $\binom{n-1}{k} c(k) = \gamma(k)/n$ and observe that $|\{R: P_{-i}^R = S\}| = (|S|)! (n-1-|S|)!/n!)$ and proceed with the proof of the main theorem of this section.

Proof of Theorem 2.5: It is easily verified that for each probability measure $\lambda$ on $[0,1]$ $\psi_\lambda$ is a semi value on G. Now, let $\psi$ be a semi value on G.

To complete the proof it suffices to establish that $\psi$ is of the form $\psi_\lambda$ for some probability measure $\lambda$ on $[0,1]$. Let $i \in U$ be fixed. For each finite subset $N$ of $U \setminus \{i\}$, $\psi$ induces a semi value on $G^{N \cup \{i\}}$. By lemma 2.6*, $\psi$ induces a probability measure $c_N$ on the subsets of $N$. If $N \subset \overline{N}$, then by considering the natural embedding of $G^N$ into $G^{\overline{N}}$, we have

$c_N(|S|) \equiv c_N(S) = \sum\limits_{S \subset T \subset (N \backslash N) \cup S} c_{\overline{N}}(T)$. Let $N_k$ be an increasing sequence of finite subsets of $U \backslash \{i\}$. The measures on the subsets of $N_k$, $k = 1,2,\dots$ are "consistent", and therefore by Kolmogorov's consistency theorem, ([12], pp. 94), there is a sequence of random variables $Y_j$, $j \in \cup N_k$ such that:

$$(2.11) \qquad \qquad \text{Prob}(Y_j = 1) + \text{Prob}(Y_j = 0) = 1$$

$$(2.12) \quad c_N(|S|) = \text{Prob}(Y_j = 1 \text{ for } j \in S \text{ and } Y_j = 0 \text{ for } j \in N\backslash S).$$

Thus $Y_j$ is an exchangeable sequence of random variables. De Finetti's theorem ([11], sec. 9.6.1) asserts that the distribution of every exchangeable infinite sequence or random variables is a mixture of distributions of sequences of independent identically distributed random variables. As $\text{Prob}(Y_j = 0 \text{ or } 1) = 1$,

for every finite sequence $\varepsilon_1, \dots, \varepsilon_n$ of 0's and 1's

$$(2.13) \quad \text{Prob}(Y_i = \varepsilon_i \text{ for } i = 1,\dots,n) = \int\limits_0^1 t^{\Sigma \varepsilon_i} (1-t)^{n - \Sigma \varepsilon_i} d\lambda(t)$$

for some probability measure $\lambda$ on $[0,1]$.

By lemma $(2.6)^*$, $(2.12),(2.13)$ and the definition of $\Delta v(t)$ we have for any game $v$ with support $N \cup \{i\}$

$$\psi v(\{i\}) = \sum\limits_{S \subset N} c_N(|S|) \, [v(S \cup \{i\}) - v(S)] =$$

$$= \sum\limits_{S \subset N} \text{Prob}(Y_j = 1 \; \forall \, j \in S \text{ and } Y_j = 0 \;\; \forall \, j \in N\backslash S) \cdot [v(S \cup \{i\}) - v(S)]$$

$$= \sum\limits_{S \subset N} (\int\limits_0^1 t^{|S|} (1-t)^{|N\backslash S|} d\lambda(t)) \, [v(S \cup \{i\}) - v(S)]$$

$$= \int\limits_0^1 E(\Delta v(t)|X_i = t) \cdot d\lambda(t).$$

It is obvious from the axiom of symmetry that the mixing measure $\lambda$ depends neither on the particular player $i$, nor on the sequence $N_k$, and thus $\lambda$ is determined by $\psi$ alone. Consequently (2.14) is valid for every game $v$ in $G$ and every player $i$ in $U$.

We proceed in order to prove that the map $\lambda \to \psi$ is 1-1. Let $\lambda_1 \neq \lambda_2$ be two different probability measures on $[0,1]$. Without loss of generality we may assume that there exists $t_1 < t_2$ with $\lambda_1([0,t_1]) > \lambda_2([0,t_2])$. Let $t = \dfrac{t_1+t_2}{2}$ and for each $n$ let $N$ be a subset of $U$ with $|N| = n$ and define the game $v_n$ by:

$$v_n(S) = \min(|S \cap N|, [t \cdot n])$$

Let $i \in N$, then by the weak law of large number $\liminf \psi_{\lambda_1} v_n(\{i\}) \geq \lambda_1([0,t_1])$ and $\limsup \psi_{\lambda_2} v_n(\{i\}) \leq \lambda_2([0,t_2])$ which proves that $\psi_{\lambda_1} \neq \psi_{\lambda_2}$. This completes the proof of the theorem.

Q.E.D.

Another plausible axiom to impose on a semi-value is continuity (in the total bounded variation norm). In order to state our characterization of the continuous semi-values we introduce the following notation. Let $W$ be the subset of $L_\infty = L_\infty(0,1)$ of all nonnegative functions $g$ with $\int_0^1 g(t) \cdot dt = 1$.

Proposition 2.15: There is a linear isometry of $W$ onto the family of continuous semi values. This isometry carries the element $g$ of $W$ to the semi value $\psi_g$, defined by

$$\psi_g v(\{i\}) = \int_0^1 g(t) \cdot E(\Delta v(t)|X_i = t) \cdot dt$$

Proof. Let $\psi$ be a continuous semi value. By theorem 2.5 there exists a probability measure $\lambda$ on $[0,1]$ such that $\psi = \psi_\lambda$. We will prove now that

if there exists a borel set $B$ with $\lambda(B) > k \cdot \ell(B)$ where $\ell$ stands for lebesque measure then $\|\psi_\lambda\| \geq k$. First assume that $B$ is an open interval $B = (t,s)$. Define the game $v^n$ on $N$ (with $|N| = n$) by:

$$v_B^n(S) = f_B^n(|S \cap N|)$$

where $f_B^n(0) = 0$, $f_B^n(k) - f_B^n(k-1) = 1$ if $k/n \in B$ and $f_N^n(k) - f_B^n(k-1) = 0$ otherwise. Observe that

$$\| v^n \|/n \xrightarrow[n \to \infty]{} s-t = \ell(B).$$

On the other hand using the weak law of large number for i.i.d. random variables

$$\underline{\lim} \frac{\|\psi_\lambda v^n\|}{n} \geq k$$

which completes the proof in case B is an open interval. Second assume that $B$ is an open set i.e., $B = \cup B_i$ where $B_i$ are open intervals. The same arguments applied to the games $v_B^n = \sum v_{B_i}^n$ yield the conclusion. For proving the general case use the fact that for every $\epsilon > 0$ there exists an open set $\overline{B} \supset B$ with $\ell(\overline{B}\backslash B) < \epsilon$. Hence if $\psi_\lambda$ is continuous there exists $k$ such that for every Borel set $B$, $\lambda(B) \leq k \cdot \ell(B)$, which implies that $\lambda$ is of the form $\lambda(B) = \int_B g(t) \cdot dt$ where $g \in L_\infty$. As $\lambda$ is a probability measure $\int_0^1 g(t) \cdot dt = 1$ and $g \in L_\infty^+$, i.e. $g \in W$.

We have now to prove that $\|g\| = \|\psi_g\|$. Actually we had already proved that $\|\psi_g\| \geq \|g\|$. To prove the other inequality we observe that

$$\frac{1}{n} \cdot \gamma^n(k) = \int_0^1 \binom{n-1}{k} t^k (1-t)^{n-1-k} \, g(t) \cdot dt$$

$$\leq \left( \int_0^1 \binom{n-1}{k} t^k (1-t)^{n-1-k} \cdot dt \right) \|g\|_\infty$$

$$= \frac{1}{n} \cdot \|g\|_\infty$$

As $\|\psi_\lambda\| = \sup\limits_{n,k} \gamma^n(k)$ we find that $\|\psi_\lambda\| \leq \|g\|_\infty$ which completes the proof of the proposition.

## 3. Definition of Semi Values for Non-Atomic Games

All definitions and notations are according to [3]. Let $(I,C)$ be a measurable space, isomorphic to $([0,1],B)$, where $B$ is the $\sigma$-field of Borel subsets of $[0,1]$, and let BV be the space of bounded variation set functions on $(I,C)$. The space of all bounded, finitely additive set functions is denoted FA, and its subspace of all non-atomic measures is denoted NA. Denote by $G$ the group of automorphisms of $(I,C)$. For each $\Theta \in G$, $\Theta^*: BV \to BV$ is defined by $\Theta^* v(S) = v(\Theta S)$. If $Q \subset BV$ then $Q^+$ denotes the subset of $Q$ of all monotonic set functions. A subset $Q$ of BV is _symmetric_ if for each $\Theta \in G$, $\Theta^* Q \subseteq Q$. An operator $\psi: Q \to BV$ is called _positive_ if, $\psi(Q^+) \subset BV^+$, and _symmetric_ if for each $\Theta \in G$, $\Theta^* \psi = \psi \Theta^*$. The members of $I$ are called _players_, the members of $C$ _coalitions_, and set functions are called _games_.

Let $Q$ be a linear symmetric subspace of BV. A _semi value_ on $Q$ is an operator $\psi$ from $Q$ into FA such that:

(3.1) $\psi$ is linear

(3.2) $\psi$ is symmetric

(3.3) $\psi$ is positive

(3.4) if $v \in Q \cap FA$ than $\psi v = v$.

## 4.  Basic Properties of Semi Values

In this section we state several general results (concerning semi values) which are immediate consequences of known results (concerning values).

Proposition 4.1:  Any semi value on a closed[1] reproducing[2] space  Q  is continuous[3].

Proof.  By proposition 4.15 of [3] any positive linear operator from  Q into  BV  is continous.
<div align="center">Q.E.D.</div>

Proposition 4.2:  Continuous semi values are diagonal[4].

Proof.  The proof in [8] that continuous values are diagonal does not make use of the efficiency axiom and therefore the same proof works here.
<div align="right">Q.E.D.</div>

Proposition 4.3:  Semi values on closed reproducing spaces are diagonal.

Proof.  Follows from propositions 4.1 and 4.2.
<div align="center">Q.E.D.</div>

Proposition 4.4:  Let  Q  be a symmetric subspace of  BV,  and let  $\psi$  be a semi value on  Q.  If  $\mu \in NA^+$  and  f  is defined on the range of  $\mu$  such that  $f \circ \mu \in Q$,  then  $\psi(f \circ \mu) = a \cdot \mu$  for some  $a \in R$.

Proof.  Follows from the proof of proposition 6.1 in [3].
<div align="center">Q.E.D.</div>

## 5.  Characterization of the Semi Values on  pNA

In this section we will characterize all the semi values on  pNA--the closed subspace of BV  spanned by all powers of  $NA^+$  measures.  This space plays a very important role in the theory of non-atomic games, and it contains many games of

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[1]Definition of the norm in  BV  appears in [3] pp. 19.

[2]Q  is reproducing if  $Q = Q^+ - Q^+$.

[3]$\psi$: Q $\rightarrow$ BV  is continuous if there exists  K  with  $\|\psi v\| \leq K \|v\|$  for all  $v \in Q$.

[4]Definition of diagonality appears in [1] pp. 252.

interest[1]. For example, pNA contains all "vector measure games" obeying
appropriate differentiability conditions, i.e. all set functions of the form
$f \circ \mu$, where $\mu = (\mu_1, \ldots, \mu_n)$ is a non-atomic finite-dimensional vector
measure and $f$ is an appropriately differentiable real valued function defined
on the range of $\mu$, with $f(0) = 0$. As our main theorem in this section uses
the notation and terminology of the "extension" we restate here the relevant
notations, definitions and results of [3] concerning the "extension". $I$ denotes
the family of all measurable function from $(I, C)$ to $([0,1], B)$. There is a
partial order on $I$; $f \geq g$ if $f(s) \geq g(s)$ for all $s \in I$. A real valued function
$w$ on $I$ with $w(0) = 0$ is called an _ideal set function_; it is called _monotonic_
if $f \geq g \Rightarrow w(f) \geq w(g)$. The characteristic function of a member $S$ of $C$ is
denoted $\chi_S$. We will sometimes denote $\chi_S$ by $S$ and $t \cdot \chi_I$ by $t$.

We are now ready to state

Theorem 5.1: (Theorem G of [3]). There is a unique mapping that associates
with each $v \in pNA$ an ideal set function $v^*$, so that

(5.2) $(\alpha v + \beta w)^* = \alpha \cdot v^* + \beta \cdot w^*$

(5.3) $(v \cdot w)^* = v^* \cdot w^*$

(5.4) $\mu^*(f) = \int_I f \cdot d\mu$

(5.5) $v$ monotonic $\rightarrow v^*$ monotonic

whenever $v, w \in pNA$, $\alpha, \beta \in R$, $\mu \in NA$, $f \in I$.

Denote $\partial v^*(t, S) = \dfrac{d}{d\tau} v^*(t\chi_I + \tau \cdot \chi_S)\big|_{\tau=0}$ then by theorem $H$ of [3] we
know that for each $v \in pNA$ and each $S \in C$, the derivative $\partial v^*(t, S)$ exists

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[1]
See, for example, Theorem B, C and J of [3].

for almost all  t  in  [0,1],  and is integrable over  [0,1]  as a function of

t,  i.e.   $\partial v^*(t,S) \in L_1(dt)$.

Denote by  W  the subset of  $L_\infty([0,1],dt) = L_\infty$  of functions  $g \in L_\infty$  with

$0 \leq g$  and  $\int_0^1 g(t) \cdot dt = 1$.

Theorem 5.6:  For each  $g \in W$  the operator  $\psi_g$: pNA → FA  defined by

(5.7)  $\psi_g v(S) = \int_0^1 g(t) \cdot \partial v^*(t,S) \cdot dt$

is a semi value.  Moreover any semi value on  pNA  is of that form.

The map  $g \to \psi_g$  of  W  onto the family of semi values on  pNA  is a linear

isometry  $\|\psi_g\| = \|g\|_{L_\infty}$.

Proof.  Let  $g \in L_\infty$  be given.  For  $v \in$ pNA,  lemma 23.1 of [3] asserts that

$\int_0^1 |\partial v^*(t,S)| \cdot dt \leq \|v\|$   hence  $|\psi_g v(s)| = |\int_0^1 g(t)\partial v^*(t,S) \cdot dt| \leq \|g\|_{L_\infty} \cdot \|v\|$

which proves that  $\psi_g v$  is bounded.  If  $S,T \subset I$  with  $S \cap T = \emptyset$  then

$\partial v^*(t, T \cup S) = \partial v^*(t,T) + \partial v^*(t,S)$  for almost all  $t^1$.  Therefore  $\psi_g v(S \cup T) =$

$\psi_g v(S) + \psi_g(T)$  which proves that  $\psi_g$  maps  pNA  into  FA.  Let  $v \in$ pNA$^+$  then

$v^*$  is also monotonic, hence  $\partial v^*(t,S) \geq 0$  and thus  $\psi_g v$  is monotonic, which

proves the positivity of  $\psi_g$.  Linearity of  $\psi_g$  follows from the linearity of the

extension as well as that of the derivative.  Symmetry of  $\psi_g$  follows from the fact

that  $\partial(\Theta^* v)^*(t,S) = \partial v^*(t,\Theta S)$  and thus  $\Theta^* \psi_g v(S) = \int g \cdot \partial v^*(t,\Theta S) \cdot dt =$

$\int g \cdot \partial(\Theta^* v)^*(t,S) \cdot dt = \psi_g(\Theta^* v)(S)$.  Finally, for  $u \in$ FA $\cap$ pNA = NA (pNA $\subset$ AC

([3] Corollary 5.3), but every member of  AC  is continuous ([3] p. 205) which

implies  $\sigma$-additivity.), $\partial \mu^*(t,S) = u(S)$  and as  $\int g \cdot dt = 1$,  $\psi_g u = u$  and this

completes the proof that  $\psi_g$  is a semi value.

Now, let  $\psi$  be a semi value on  pNA.  Let  $\mu$  be a fixed probability measure

in  NA.  Each  $f \in L_1$  induces a game  $v_f$  defined by

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[1]It follows for instance from Proposition 24.1 of [3], which states a much
stronger result.

$$v_f(S) = \int_0^{\mu(S)} f(t) \cdot dt \ .$$

In other words, $f$ defines a function $F: [0,1] \to R$ by $F(s) = \int_0^s f(t) \cdot dt$, and $v = F \circ \mu$. As $f \in L_1$, $F$ is absolutely continuous and therefore $v \in pNA$. From proposition 4.4 it follows that $\psi v_f = C(f) \cdot \mu$ where $C(f)$ is a constant (which is independent of $\mu$). Observe that $v_{f+g} = v_f + v_g$ and thus the linearity of $\psi$ implies that $C$ is linear. We proceed now in order to show that $C$ is continuous. Observe that $\|v\| = \|f\|_{L_1}$. Now, as $pNA$ is internal (Propositon 7.19 of [3]), it is in particular a closed reproducing space and thus by proposition 4.3 $\psi$ is continuous on $pNA$, i.e., there exists a constant $K$ with $\|\psi v\| \leq K \cdot \|v\|$, which in particular implies that $|C(f)| = \|C(f) \cdot \mu\| \leq K \cdot \|v_f\| = K \cdot \|f\|_{L_1}$, which proves that $C: L_1 \to R$ is continuous. Thus $C: L_1 \to R$ is a linear continuous function and therefore is of the form $C(f) = \int_0^1 f \cdot g \cdot dt$ for some $g \in L_\infty$. Now, let $g \in L_\infty$ be associated with the function $C$ induced by the semi value $\psi$. We shall continue in order to show that $\psi = \psi_g$, where $\psi_g$ is defined by (5.7). As was shown in the beginning of the proof $\psi_g(pNA) \subset FA$ and $|\psi_g v(S)| \leq \|g\|_{L_\infty} \cdot \|v\|$ which implies that $\psi_g$ is continuous. For each $f \in L_1$, $\partial v_f^*(t,S) = f(t) \cdot \mu(S)$ for almost all $t$, and thus $\psi_g v_f(S) = \mu(S) \cdot \int f \cdot g \cdot dt = C(f) \cdot \mu(S) = \psi v_f(S)$ and therefore $\psi_g v_f = \psi v_f$. The linear symmetric subspace spanned by the games $v_f$, $f \in L_1$ is dense in $pNA$ (it contains all power of $NA$ measures), but $\psi$ and $\psi_g$ are linear and symmetric and thus coincide on this span and, as they are also continuous, they coincide on $pNA$. We have now to show that $g \in W$. For $v \in NA \subset FA \cap pNA$ $\partial v^*(t,S) = v(S)$ and thus $\psi_g v(S) = (\int_0^1 g(t) \cdot dt) v(S)$ which shows that $\int_0^1 g(t) \cdot dt = 1$. Let $B_\epsilon = \{t: g(t) \leq - \epsilon\}$ and let $f$ be the characteristic function of $B_\epsilon$, then $f \geq 0$ and hence $v_f$ is monotonic, but as $\psi_g v_f(I) = \int f \cdot g \cdot dt \leq - \epsilon \cdot \lambda(B_\epsilon)$ ($\lambda$ denotes the lebesque measure on $[0,1]$),

and $\psi_g = \psi$ is positive, $\lambda(B_\varepsilon) = 0$. As this holds for any $\varepsilon > 0$, $g \geq 0$, this completes the proof that any semi value $\psi$ is $\psi_g$ for some $g \in W$.

Now for $g \in L_\infty$ there exists $f \in L_1$ with $\|f\|_{L_1} = 1$ and $\int f \cdot g \cdot dt = \|g\|_{L_\infty}$. Observe that $\|v_f\| = \|f\|_{L_1} = 1$ and that $\|\psi_g v_f\| = \|g\|_\infty$ hence $\|\psi_g\| \geq \|g\|_\infty$, on the other hand for $v \in pNA^+$ $\|\psi_g v\| = \psi_g v(I) = \int g(t) \cdot \partial v^*(t,I) \cdot dt \leq \|g\|_\infty$ $\cdot \int_0^1 \partial v^*(t,I) \cdot dt = \|g\|_\infty \cdot \|v\|$. In the general case, when $v$ is not necessarily monotonic, let $\varepsilon > 0$ be given, and set $v = u - w$ where $u$ and $w$ are in $pNA^+$ and $\|v\| + \varepsilon \geq \|u\| + \|w\|$; such $u$ and $w$ exists by the internality of pNA (proposition 7.19 of [3]). $\|\psi_g v\| \leq \|\psi_g u\| + \|\psi_g w\| \leq \|g\|_{L_\infty} (\|u\| + \|w\|)$ $\leq \|g\|_{L_\infty} (\|v\| + \varepsilon)$ and if we let $\varepsilon \to 0$, $\|\psi_g v\| \leq \|g\|_{L_\infty} \cdot \|v\|$ which completes the proof of the equality $\|\psi_g\| = \|g\|_{L_\infty}$. This completes the proof of theorem 5.6.

Q.E.D.


## 6. Characterization of the Semi Values on a Class of Political Economic Games

In the purely economic situation, we are usually encountered with games in pNA (or in pNAD)[1], but in many political economic situations we face games of the form $v = u \cdot q$ where $q$ is in pNA and $u$ is a jump function with respect to a given NA probability measure $\mu$, i.e.,

$$u(S) = \begin{cases} 1 & \text{if } \mu(S) \geq \alpha \\ 0 & \text{if } \mu(S) < \alpha \end{cases}.$$

Such games arose for instance in models for taxation (See Aumann-Kurz [1] and [2]). We denote by u*pNA the minimal linear symmetric space containing pNA and all games of the form $u \cdot q$, $q \in pNA$.

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[1] For definition see [3], p. 253.

**Theorem 6.1:** For any pair $(a,g)$, $a \in R^1$ and $g \in W$, there is a semi value $\psi(a,g)$ on u*pNA such that for any $q \in$ pNA

(6.2)
$$\psi_{(a,g)}q(S) = \int_0^1 g(t) \cdot \partial q^*(\tau,S) \cdot dt$$

and

(6.3)
$$\psi_{(a,g)}(u \cdot q)(S) = a \cdot q^*(\alpha) \cdot \mu(S) + \int_\alpha^1 g(t) \cdot \partial q^*(t,S) \cdot dt$$

Moreover, any continuous semi value on u*pNA is of that form. The mapping $(a,g) \to \psi_{(a,g)}$ is 1-1 and $\|\psi_{(a,g)}\| = \max(a, \|q\|_{L_\infty})$. The proof of the theorem is accomplished in several stages. First we shall start and prove a general lemma concerning the range of a vector of members of pNA. This is a generalization of a result of Dvoretzky, Wald and Wolfowitz ([6], p. 66, Theorem 4).

**Lemma 6.4:** Let $\nu$ be a finite dimensional vector of measures in NA, and let $m$ be a positive integer. Then for each $m$-tuple $f_1,\ldots,f_m$ of ideal sets such that $f_1 + \ldots + f_m = 1 = X_I$, and each $k$ tuple $q_1,\ldots,q_k$ of members of pNA, and each $\epsilon > 0$ there is a partition $(T_1,\ldots,T_m)$ of $I$, $T_i$ in $C$ such that for all $A \subset (1,\ldots,m)$ and all $1 \le j \le k$

$$\nu(\underset{i \in A}{\cup} T_i) = \int (\underset{i \in A}{\sum} f_i) \cdot d\nu$$

and

$$|q_j(\underset{i \in A}{\cup} T_i) - q_j^*(\underset{i \in A}{\sum} f_i)| < \epsilon.$$

**Remark:** The same result holds if pNA is replaced by pNA' (replace in the proof $\| \ \|$ by $\| \ \|'$).

**Proof.** From the definition of pNA it follows that for each $1 \le j \le k$ there exist a polynom $v_j$ of NA-measures; $v_j = P_j(\mu_1^j,\ldots,\mu_{n_j}^j)$ with $\|q_j - v_j\| < \epsilon$. By theorem 4 of [6] there is a partition $(T_1,\ldots,T_m)$ of $I$, $T_i$ in $C$ such that for all $1 \le i \le m$, $1 \le j \le k$, and $1 \le \ell \le n_j$

$$\nu(T_i) = \int f_i \, d\nu$$

and

$$\mu_\ell^j(T_i) = \int f_i \cdot d\mu_\ell^j \quad .$$

From the finite additivity of members of NA, we deduce that for each

$A \subset \{1,\ldots,m\}$, $1 \leq j \leq k$ and $1 \leq \ell \leq n_j$ $\nu(T(A)) = \int f(A) \cdot d\nu$ and

$\mu_\ell^j(T(A)) = \int f(A) \cdot d \mu_\ell^j$ where $T(A) = \underset{i\in A}{\cup} T_i$ and $f(A) = \underset{i\in A}{\sum} f_i$.

From the last equalities and theorem 5.1 it follows that also $v_j(T(A)) = v_j^*(f(A))$.

Thus

$$\left| q_j(T(A)) - q_j^*(f(A)) \right| \leq \left| g_j(T(A)) - v_j(T(A)) \right| + \left| v_j(T(A)) - v_j^*(f(A)) \right| +$$

$$+ \left| v_j^*(f(A)) - q_j^*(f(A)) \right| \leq \| q_j - v_j \| + 0 + \| v_j^* - q_j^* \|$$

and as $\| v^* \| = \| v \|$ for each $v \in pNA$ $\left| q_j(T(A)) - q_j^*(A)) \right| < 2\epsilon$. This

completes the proof of lemma 6.4.

<div align="center">Q.E.D.</div>

We will proceed in order to show that (6.2) and (6.3) define a unique

linear symmetric operator from u*pNA into FA. For this we shall need the

following lemma.

__Lemma 6.5:__ If $w = v + \sum_{i=1}^{n} \theta_i^* u \cdot q_i$ is monotonic, where $v \in pNA$,

$q_i \in pNA$ and $\theta_i \in G$ $i = 1,\ldots,n$, then for any $S \in C$ and $g \in L_\infty$, $g \geq 0$

$$(6.6) \qquad \int_0^\alpha g(t) \cdot \partial v^*(t,S) \cdot dt \geq 0$$

(6.7) $\quad \int\limits_\alpha^1 g(t) \cdot \partial v^*(t,S) \cdot dt + \sum\limits_{i=1}^n \int\limits_\alpha^1 g(t) \cdot \partial q_i^*(t,\Theta S) \cdot dt \geq 0$

and

(6.8) $\qquad\qquad\qquad \sum\limits_{i=1}^n \mu(\Theta_i S) \cdot q_i^*(\alpha) \geq 0 .$

Proof. Assume that $w$ is monotonic. For proving (6.6) it is enough to show that for each $0 < t < \alpha$ for which $\partial v^*(t,S)$ is defined, $\partial v^*(t,S) \geq 0$. Let $0 < t < \alpha$, and let $0 < h < \alpha - t$; then $t + hS \leq t + h < \alpha$ and therefore $(\Theta_i^* \mu)^*(t + hS) < \alpha$ for each $i = 1,\ldots,n$. Let $\varepsilon > 0$ be given. By lemma 6.4 for any $\varepsilon > 0$, there are $T_1$ and $T_2$ in $C$ with $T_2 \supset T_1$ such that $\Theta_i^* \mu(T_1) = (\Theta_i^* \mu)^*(t)$, $(\Theta_i^* \mu)(T_2) = (\Theta_i^* \mu)^*(t + hS)$ and $|v^*(t) - v(T_1)| < \varepsilon$ and $|v^*(t+hS) - v(T_2)| < \varepsilon$. Therefore $w(T_2) - w(T_1) =$ $= v(T_2) - v(T_1) \leq v^*(t+hS) - v^*(t) + 2\varepsilon$. As $w$ is montonic, $v^*(t+hS) - v^*(t) \geq$ $\geq -2\varepsilon$, and as this holds for any $\varepsilon > 0$, $v^*(t+hS) - v^*(t) \geq 0$ and therefore $\partial v^*(t,S) \geq 0$ for any $0 < t < \alpha$ for which $\partial v^*(t,S)$ is defined. This completes the proof of (6.6).

A similar argument proves (6.7) and therefore we omit the proof. For proving (6.8) assume for the moment that $\Theta_i^* \mu \neq \Theta_j^* \mu$ for $1 \leq i < j \leq n$. In such a case there is $T$ in $C$ with $\Theta_i^* \mu(T) \neq \Theta_j^* \mu(T)$ for $1 \leq i < j \leq n$. Let $1 \leq k \leq n$ be given. Denote $a = \Theta_k^* \mu(T)$. For any $0 < \beta < 1$ and $0 < \gamma < \alpha$ consider the ideal sets $g(\gamma) = a + \gamma(T-a)$ and $\beta \cdot g(\gamma)$. Observe that $(\Theta_i^* \mu)^*(g(\gamma)) = \alpha$ if and only if $i = k$. Thus, applying lemma 6.4 with the vector measure $(\Theta_1^* \mu,\ldots,\Theta_n^* \mu)$ and $v, \Theta_1^* q_1,\ldots,\Theta_n^* q_n$ in pNA, to the ideal sets $\beta \cdot g(\gamma)$, $g(\gamma)$, and letting $\beta \to 1$ we conclude that $(\Theta_k^* q_k)^*(\alpha + \gamma T - \gamma\alpha) \geq 0$ and if we let $\gamma \to 0$ we deduce that $q_k^*(\alpha) \geq 0$. In the general case, define the equivalence relation $\sim$ on $\{1,\ldots,n\}$ by $i \sim j$ iff, $\Theta_i^* \mu = \Theta_j^* \mu$. Then the same argument yield that if $A$ is an equivalence class then $\sum\limits_{i \in A} q_i^*(1/\alpha) \geq 0$

and therefore (6.8) is proved. This completes the proof of lemma 6.5.

<div align="right">Q.E.D.</div>

**Lemma 6.9:** Let $g$ be in $W$ and $a$ in $R^\perp$. Then (6.2) and (6.3) defines (uniquely) a semi value $\psi_{(a,g)}$ on $u*pNA$.

**Proof.** Any element $w$ in $u*pNA$ is of the form $w = v + \sum_{i=1}^{n} \theta_i^*(u \cdot q_i)$, $\theta_i \in G$, $v$, $q_i \in pNA$. By linearity and symmetry, it follows from (6.2) and (6.3) that

$$(6.10) \quad \psi_{(a,g)}w(S) = \int_0^1 g(t) \cdot \partial v^*(t,S) \cdot dt + \sum_{i=1}^{n} \int_{1/\alpha}^1 g(t) \cdot \partial q_i^*(t,\theta_i S) \cdot dt$$

$$+ \sum_{i=1}^{n} a \cdot q_i^*(1/\alpha) \cdot \theta_i^* \mu(S).$$

We have to show that $\psi_{(a,g)}$ is well defined, i.e., that it is independent of the representation of $w$. Because of the linearity it is enough to show that if $w = 0$ then $\psi_{(a,g)}w = 0$. If $w = 0$ then by lemma (6.5) we conclude that $\psi_{(a,g)}w(S) \geq 0$, and that $\psi_{(a,g)}(-w)(S) = -\psi_{(a,g)}w(S) \geq 0$ which means that $\psi_{(a,g)}w = 0$. Linearity and symmetry of $\psi_{(a,g)}$ follows from the definition. The finite additivity of $\partial q^*(t,S)$ ($q \in pNA$) as well as that of $\theta_i^* \mu$ implies that $\psi_{(a,g)}w$ is finitely additive. Positivity of $\psi_{(a,g)}$ follows now from lemma (6.5) and the finite additivity of $\psi_{(a,g)}w$. Obviously $u*pNA$ is reproducing; hence the positivity of $\psi_{(a,g)}$ and the finite additivity of $\psi_{(a,g)}w$ implies that $\psi_{(a,g)}w$ is in FA whenever $w$ is in $u*pNA$. Now let $w \in (u*pNA) \cap FA$. We have to show that $\psi_{(a,g)}w = w$. Without loss of generality, we may assume that $w = v + \sum_{i=1}^{n} \theta_i^* \cdot u \cdot q_i$ where $v \in pNA$ and, $q_i \in pNA$ and $\theta_i^* \mu = \theta_j^* \mu$ iff $i = j$. First we shall show that $q_i^*(\alpha) = 0$ for each $i$, $1 \leq i \leq n$. Let $1 \leq k \leq n$ be given, and define $g(\gamma)$ as in the proof of (6.8) in lemma 6.5. Applying lemma 6.4 to the 3-tuple $(1-\beta) g(\gamma)$, $\beta g(\gamma)$, $1-g(\gamma)$ with the vector measure $(\theta_1^* \mu, \ldots, \theta_n^* \mu)$ and $v$, $\theta_1^* q_1, \ldots, \theta_n^* q_n \in pNA$, and using the finite additivity

of $w$, and the fact that $\hat{v}*((1-\beta)g) \xrightarrow[\beta \to 1]{} 0$, we deduce (by letting $\beta \to 1$) $(\Theta_k^* q_k)^*(g(\gamma)) = 0$ and if we let $\gamma \to 0$ we conclude that $q_k^*(\alpha) = 0$. Let $S$ be in $C$, with $\mu(\Theta_i S) < \alpha$. In that case $w(S) = v(S)$, and by using the finite additivity of $w$ and lemma 6.4, we see that $v^*(hS) = h(v(S))$ for any rational $0 \leq h \leq 1$ and then by continuity of $v^*$ we deduce that $v^*(hS) = h \cdot v(S)$ for any real $h$, $0 \leq h \leq 1$. Therefore $\partial v^*(0,S) = v(S)$. Now, let $0 < t < \alpha$,. and let $S \in C$ be given. Again using lemma 6.4 to the vector measure $\Theta_i^* \mu$ $1 \leq i \leq n$, and the game $v \in pNA$ and the 3-tuple $hS$, $t$, $1-t-hS$ $h < \alpha-t$ we have for any $\varepsilon > 0$ a partition $(T_1, T_2, T_3)$ of $I$ with $|v(T_1) - v^*(hS)| < \varepsilon$, $|v(T_2) - v^*(t)| < \varepsilon$ and $|v(T_1 \cup T_2) - v^*(t+hS)| < \varepsilon$ and $\Theta_1^* \mu(T_1 \cup T_2) < \alpha$. Hence $w(T_1 \cup T_2) = v(T_1 \cup T_2)$, $w(T_1) = v(T_1)$ and $w(T_2) = v(T_2)$. Therefore, using the finite additivity of $w$ we have $v(T_1 \cup T_2) - v(T_2) = v(T_1) - v(\emptyset) = v(T_1)$, and as $|v(T_1 \cup T_2) - v^*(t+hS)| < \varepsilon$, $|v(T_2) - v^*(t)| < \varepsilon$ and $|v(T_1) - v^*(hS)| < \varepsilon$ $|[v^*(t+hS) - v^*(t)] - v^*(hS)| < 3\varepsilon$ and as this holds for any $\varepsilon > 0$, $v^*(t+hS) - v^*(t) = v^*(hS) = h \cdot v(S)$ and therefore $\partial v^*(t,S)$ exists and equals $v(S)$. In a similar way, by using lemma 6.4 to the vector measure $\Theta_i^* \mu$, $1 \leq i \leq n$, and the games $v \in pNA$, $\Theta_i^* q_i$ $1 \leq i \leq n$ and the 3 tuple $hS$, $t$, $1-t-hS$, $h < 1-t$ we can prove that for $\alpha < t < 1$ $\partial(\sum_{i=1}^n \Theta_i^* q_i)^*(t,S) = v(S)$. Therefore as $\int_0^1 g(t)dt = 1$ we conclude that $\psi_{(a,g)} w(S) = v(S) = w(S)$ whenever $S$ is in $C$ with $\mu(\Theta_i S) < \alpha$. For $S$ in $C$ there exists always a partition $S = S_1 \cup \ldots \cup S_k$ with $S_i$ $i = 1, \ldots, k$ in $C$ and $\mu(\Theta_i S_j) < \alpha$ $1 \leq i \leq n$, $1 \leq j \leq k$. Therefore by the finite additivity of $w$ as well as that of $\psi w$ we have $\psi_{(a,g)} w(S) = \sum_{i=1}^k \psi_{(a,g)} w(S_i) = \sum_{i=1}^k w(S_i) = w(S)$ which completes the proof of lemma 6.9.

Lemma 6.10: Let $g$ be in $W$ and $a$ in $R^1$. Then the semi value $\psi_{(a,g)}$ on $u*pNA$ defined by (6.2) and (6.3) is continuous and $\|\psi_{(a,g)}\| = \max\{a, \|g\|_{L_\infty}\}$.

$\underline{\text{Proof}}$. Let $w$ be in $u*pNA$. Without loss of generality we may assume

that $w = v + \sum_{i=1}^{n} \Theta_i^*(u \cdot q_i)$ with $v$ in $pNA$, $q_i \in pNA$ $(1 \leq i \leq n)$, and

$\Theta_i \in G$ with $\Theta_i^* \mu = \Theta_j^* \mu$ iff $i = j$ $(1 \leq i, j \leq n)$. As mentioned in the proof

of lemma 6.5 there exists $T$ in $C$ with $\mu(\Theta_i T) = \mu(\Theta_j T)$ iff $i = j$. We may

assume that $\mu(\Theta_i T) > \frac{\alpha+1}{2}$ for each $1 \leq i \leq n$ (by applying theorem 4 of [6]

p. 66 for instance to the vector measure $\Theta_1^* \mu, \ldots, \Theta_u^* \mu$ and the ideal set

$\frac{3+\alpha}{4} + \frac{1-\alpha}{4} \cdot T$). Let $T_1$, $T_2$ be disjoint sets in $C$ with $T_1 \cup T_2 = I$.

For each $0 < \beta < 1$ choose an integer $k$ with $2/k < \beta(1-\alpha)/2$ and choose $m$

with $1/m < \alpha \cdot \min_{i \neq j} |\mu(\Theta_i T) - \mu(\Theta_j T)|$. Let $\ell$ be the largest even number which

is smaller than $2\alpha k$. Consider the $2k + m + 1$ tuple $f_1, \ldots, f_{2k}$, $g_1, \ldots, g_m$, $h$

where $f_i = \frac{1-\beta}{k} T_1$ whenever $i$ is odd and $f_i = \frac{1-\beta}{k} T_2$ whenever $i$ is even

$(1 \leq i \leq 2k)$, $g_i = \frac{1}{m} \beta T$ $(1 \leq i \leq m)$ and $h = \beta(1-T)$. Apply to that

$2k + m + 1$ tuple lemma 6.4 with the vector measure $(\Theta_1^* \mu, \ldots, \Theta_n^* \mu)$ and the

members $v$, $\Theta_i q_i \in pNA$ $(1 \leq i \leq n)$ and $\epsilon = \frac{1}{(k+m)^2}$. Then look on the chain

$\Omega$ associated by lemma 6.4 to the chain

$$0 \leq f_1 \leq f_1 + f_2 \leq \cdots \leq \sum_{i \leq \ell} f_i \leq (\sum_{i \leq \ell} f_i) + g_1 \leq \cdots$$

$$\cdots \leq (\sum_{i=1}^{\ell} f_i) + (\sum_{i=1}^{m} g_i) \leq (\sum_{i=1}^{\ell} f_i) + (\sum_{i=1}^{m} g_i) + h \leq \cdots$$

$$\leq (\sum_{i=1}^{\ell+1} f_i) + (\sum_{i=1}^{m} g_i) + h \leq \cdots \leq \sum_{i=1}^{2k} f_i + \sum_{i=1}^{m} g_i + h = 1.$$

Note that

$$(6.12) \qquad \sum_{i=1}^{\ell} f_i = t \quad \text{where} \quad (\alpha - 2/k)(1-\beta) \leq t < \alpha(1-\beta) < \alpha$$

and

$$(6.13) \qquad \Theta_j^* \mu ( \sum_{i=1}^{\ell} f_i + \sum_{i=1}^{m} g_i ) > (\alpha - 2/k)(1-\beta) + \beta(\frac{1+\alpha}{2}) =$$

$$= \alpha + \beta(1-\alpha)/2 - 2/k > \alpha, \qquad 1 \le j \le n .$$

Observe that $\Theta_j^* \mu ( \sum_{i=1}^{\ell} f_i ) < \alpha(1-\beta)$ and that $\Theta_j^* \mu ( \sum_{i=1}^{m'} g_i ) \le \frac{m'}{m} \cdot \beta, 1 \le \ldots \le m.$
Hence if $\Theta_j^* \mu ( \sum_{i=1}^{\ell} f_i + \sum_{i=1}^{m'} g_i ) \ge \alpha$ then $m'/m \ge \alpha$. For each $1 \le i \le n$ let

$m_i$ be the minimal integer in $[1,m]$ for which $\Theta_i^* \mu ( \sum_{i=1}^{\ell} f_i + \sum_{j=1}^{m_i} g_j ) \ge \alpha.$

If $i \ne j$ and $m_i = m_j$ than $|\Theta_i^* ( \sum_{j=1}^{m_i} g_j ) - \Theta_j^* \mu ( \sum_{i=1}^{m_j} g_i )| \le \frac{\beta}{m}.$ As

$m_i = m_j \ge \alpha \cdot m$ we conclude that $|\Theta_i^* \mu (T) - \Theta_j^* (T)| \le \frac{1}{m\alpha}$ which contradicts the
assumption that $1/m < \alpha \cdot \min_{i \ne j} |\Theta_i^* \mu (T) - \Theta_j^* (T)|.$

Assume that $v$ and $q_i (1 \le i \le 1)$ are continuously differentiable, then
if we let $\beta \to 0$ we find that

$$\|w\|_\Omega \longrightarrow \int_0^\alpha |\partial v^* (t,T_1)| \cdot dt + \int_0^\alpha |\partial v^* (t,T_2)| \cdot dt +$$

$$+ \int_\alpha^1 |\partial v^* (t,T_1) + \sum_{i=1}^n \partial q_i^* (t, \Theta_i T_1)| \cdot dt +$$

$$+ \int_\alpha^1 |\partial v^* (t,T_2) + \sum_{i=1}^n \partial q_i^* (t,\Theta_i T_2)| \cdot dt +$$

$$+ \sum_{i=1}^n |q_i^*(\alpha)| .$$

As for each $w \in pNA, T \in C$ we have that $\int_0^1 |\partial w^* (t,T)| \cdot dt \le \|w\|$

we conclude that for each $w \in u*pNA$

$$\|w\| \geq \int_0^\alpha |\partial v^*(t,T_1)| \cdot dt + \int_0^\alpha |\partial v^*(t,T_2)| \cdot dt +$$

$$+ \int_\alpha^1 |\partial v^*(t,T_1) + \sum_{i=1}^n \partial q_i^*(t,\Theta_i T_1)| \cdot dt +$$

$$+ \int_\alpha^1 |\partial v^*(t,T_2) + \sum_{i=1}^n \partial q_i^*(t,\Theta_i T_2)| \cdot dt +$$

$$+ \sum_{i=1}^n |q_i^*(\alpha)|.$$

But, for $j = 1,2$

$$|\psi_{(a,g)} w(T_j)| \leq \|g\|_\infty [\int_0^\alpha |\partial v^*(t,T_j)| \cdot dt +$$

$$+ \int_\alpha^1 |\partial v^*(t,T_1) + \sum_{i=1}^n \partial q_i^*(t,\Theta_i T_j)| \cdot dt] + a \sum_{i=1}^n \mu(\Theta_i T_j) \cdot |q_j^*(\alpha)|.$$

As for any $\varepsilon \to 0$ there exists disjoint sets $T_1$ and $T_2$ in $C$ with

$|\psi_{(a,g)} w(T_1)| + |\psi_{(a,g)} w(T_2)| \geq \|\psi_{(a,g)} w\| - \varepsilon$ we conclude that

$|\psi_{(a,g)} w\| \leq \max(a, \|g\|_\infty) \cdot \|w\|$ which proves that $\|\psi_{(a,g)}\| \leq \max(a, \|g\|_\infty)$.

We omit the easy verification of the inequality $\|\psi_{(a,g)}\| \geq \max(a, \|g\|_\infty)$

which is needed to complete the proof of lemma 6.11.

Proof of Theorem 6.1: We have already seen that for $a$ in $R^1$ and $g$

in $W$ (6.2) and (6.3), define (uniquely) a semi-value $\psi_{(a,g)}$ on $u*pNA$.

Now we have to show that any continuous semi-value on $u*pNA$ is of that form.

Let $\psi$ be a continuous semi-value on $u*pNA$. In particular, $\psi$ induces a

semi-value on $pNA$ and therefore by theorem 5.6 there is $g$ in $W$ with

$$(6.14) \qquad \psi v(S) = \int_0^1 g(t) \cdot \partial v^*(t,S) \cdot dt \quad \text{for each } v \text{ in } pNA.$$

Let $\nu$ be a probability measure in NA, and $k$ a positive integer. For any $\delta > 0$, $\delta < 1/2\min(\alpha, 1-\alpha)$ define $F_\delta: [0,1] \to R^\perp$ by

$$
F_\delta(X) = \begin{cases} 0 & \text{if } |X-\alpha| \geq 2\delta \\ 1 & \text{if } |X-\alpha| \leq \delta \\ 1 - 1/\delta \ (|X-\alpha|-\delta) & \text{if } \delta < |X-1/2| < 2\delta \ . \end{cases}
$$

and define $\tilde{v}_\delta$ by:

$$
(6.15) \qquad \tilde{v}_\delta = (F_\delta \circ \nu) \cdot (F_\delta \circ \mu)(\nu^k - \mu^k)
$$

First we shall show that

$$
(6.16) \qquad \| u \cdot \tilde{v}_\delta \| \leq 32 \cdot k \cdot \delta \ .
$$

Define $U = \{S: 0 \leq \mu(S)-\alpha < 2\delta, \ |\nu(S)-\alpha| \leq 2\delta\}$, and let $\Omega: \emptyset = S_0 \subset S_1 \subset \ldots \subset S_L = I$ be a chain. $\| u \cdot \tilde{v}_\delta \|_\Omega = \sum_{i=1}^{L} |u \cdot \tilde{v}_\delta(S_i) - u \cdot \tilde{v}_\delta(S_{i-1})|$. Let $i_0$ be the first index for which $S_{i_0} \in U$ and let $j_0$ be the last index for which $S_{j_0} \in U$. Then from the definition of $U$ it follows that $S_i \in U \Longleftrightarrow i_0 \leq i \leq j_0$, and for $S \notin U \ u \cdot \tilde{v}_\delta(S) = 0$. Therefore

$$
\| u \cdot \tilde{v} \|_\Omega = \sum_{i=i_0}^{j_0+1} |u \cdot \tilde{v}_\delta(S_i) - u \cdot \tilde{v}_\delta(S_{i-1})| =
$$

$$
= |u \cdot \tilde{v}_\delta(S_{i_0})| + |u \cdot \tilde{v}(S_{j_0})| + \sum_{i=i_0+1}^{j_0} |\tilde{v}_\delta(S_i) - \tilde{v}_\delta(S_{i-1})|
$$

$$
\leq 8 \cdot k \cdot \delta + \sum_{i=i_0+1}^{j_0} |\tilde{v}_\delta(S_i) - \tilde{v}_\delta(S_{i-1})| \ .
$$

But

$$\sum_{i=i_0+1}^{j_0} |\tilde{v}_\delta(S_i) - \tilde{v}_\delta(S_{i-1})| = \sum_{i=i_0+1}^{j_0} |(F_\delta \circ \nu)(F_\delta \circ \nu)(S_i)(\nu^k - \mu^k)(S_i)$$

$$- (F_\delta \circ \nu)(F_\delta \circ \mu)(S_i) \cdot (\nu^k - \mu^k)(S_{i-1}) + (F_\delta \circ \nu)(F_\delta \circ \mu)(S_i) \cdot (\nu^k - \mu^k)(S_{i-1})$$

$$- (F_\delta \circ \nu)(F_\delta \circ \mu)(S_{i-1})(\nu^k - \mu^k)(S_{i-1}) | \leq$$

$$\leq \max_{S \in U} |(F_\delta \circ \nu)(F_\delta \circ \mu)(S)| \cdot \sum_{i=i_0+1}^{j_0} |(\nu^k - \mu^k)(S_i) - (\nu^k - \mu^k)(S_{i-1})|$$

$$+ \max_{S \in U} |(\nu^k - \mu^k)(S)| \cdot \sum_{i=i_0+1}^{j_0} |(F_\delta \circ \nu) \cdot (F_\delta \circ \mu)(S_i) - F_\delta \circ \nu)(F_\delta \circ \mu)(S_{i-1})|.$$

But $\max|(F_\delta \circ \nu)(F_\delta \circ \mu)(S)| \leq 1$ and

$$\sum_{i=i_0+1}^{j_0} |(\nu^k - \mu^k)(S_i) - (\nu^k - \mu^k)(S_{i-1})| \leq (\nu^k + \mu^k)(S_{j_0}) - (\nu^k + \mu^k)(S_{i_0}) \leq 8 \cdot k \cdot \delta$$

and

$$\max_{S \in U} |(\nu^k - \mu^k)(S)| \leq 4k\delta$$

and

$$\|(F_\delta \circ \nu) \cdot (F_\delta \circ \mu)\| \leq \|F_\delta \circ \nu\| \cdot \|F_\delta \circ \mu\| \leq 4.$$

Therefore $\sum_{i=i_0+1}^{j_0} |\tilde{v}_\delta(S_i) - \tilde{v}_\delta(S_{i-1})| \leq 1 \cdot 8 \cdot k \cdot \delta + 4 \cdot k \cdot \delta \cdot 4 = 24k\delta,$

hence $\| u \cdot \tilde{v}_\delta \|_\Omega \le 32k\delta$. As this holds for any chain $\Omega$ (6.16) is proved.

Define $G : [0,1] \to R^1$ by

$$G_\delta(X) = \begin{cases} 0 & \text{if} \quad X \ge \alpha+2\delta \\ 1 & \text{if} \quad X \le \alpha+\delta \\ 1 - \frac{1}{\delta}(X-\alpha-\delta) & \text{if} \quad \alpha+\delta < X \le \alpha+2\delta \end{cases}$$

and define $\overline{v}_\delta$ by

(6.17) $$\overline{v}_\delta = (G_\delta \circ \nu) \cdot (G_\delta \circ \mu)(\nu^k - \mu^k).$$

First observe that $\overline{v}_\delta \in pNA$ (although $(G \circ \nu)(G \circ \mu) \notin pNA$). Define $\mathcal{D}$ to be the diagonal neighborhood defined by

$$\mathcal{D} = \{S: |\mu(S) - \nu(S)| < \delta\}.$$

Let $S \in \mathcal{D}$ and denote $v = \nu^k - \mu^k$; then $u(v - \tilde{v}_\delta)(S) = (v - \overline{v}_\delta)(S)$, because if $\mu(S) < \alpha$ and $S \in \mathcal{D}$ then $\nu(S) < \alpha + \delta$ and therefore $\overline{v}_\delta(S) = v(S)$ and of course then $u \cdot (v - \tilde{v}_\delta)(S) = 0 = (v - \overline{v}_\delta)(S)$, and if $\mu(S) \ge \alpha$ then $u \cdot (v - \tilde{v}_\delta)(S) = (v - \tilde{v}_\delta)(S)$, and $\nu(S) \ge \alpha - \delta$ and for $x \ge \alpha - \delta$, $G_\delta(x) = F_\delta(x)$ which yield that $(v - \overline{v}_\delta)(S) = (v - \tilde{v}_\delta)(S)$. Thus we have seen that

(6.18) $u \cdot (v - \tilde{v}_\delta)$ coincides with $v - \overline{v}_\delta$ on a diagonal neighborhood.

As $\psi$ is continuous proposition (4.2) and (6.18) implies that

(6.19) $$\psi(u(v - \tilde{v}_\delta)) = \psi(v - \overline{v}_\delta).$$

Now we claim that

$$(6.20) \qquad \partial \overrightarrow{v}^*_\delta(t,S) = \begin{cases} 0 & \text{if } t > \alpha+2\delta \\ \partial v^*(t,S) & \text{if } t < \alpha+\delta \end{cases}$$

To prove (6.19) observe that if $t > \alpha+2+\delta$ and $h > 0$ then
$[(G_\delta o \nu) \cdot (G_\delta o \mu)(\nu^k-\mu^k)]^*(\alpha) = 0 = [(G_\delta o \nu) \cdot (G_\delta o \mu) \cdot (\nu^k-\mu^k)]^*(t+hS)$ and
if $t < \alpha+\delta$ and $h < 0$ then $[(G_\delta o \nu)(G_\delta o \mu)(\nu^k-\mu^k)^*(t+hS)$. Now, as
$v - \overline{v}_\delta$ is in pNA, (6.14) and (6.19) implies that

$$(6.21) \qquad |\psi(v-\overline{v}_\delta)(S) - \int_{\alpha+2\delta}^{1} \partial v^*(t,S) \cdot g(t) \cdot dt| \leq |\int_{\alpha+\delta}^{\alpha+2\delta} g(t) |\partial(v-\overline{v}_\delta)^*(t,S) \cdot dt| \underset{\delta \to 0}{\longrightarrow} 0$$

If we let $\delta \to 0$, (6.21),(6.19) and (6.16) implies that

$$(6.22) \qquad \psi(u \cdot (\nu^k-\mu^k))(S) = \int_{\alpha}^{1} g(t) \cdot \partial(\nu^k-\mu^k)^*(t,S) \cdot dt .$$

Observe that $u \in u*pNA$. By proposition 4.4 $\psi u = a \cdot \mu$, and by the positivity of $\psi$, $a \in R^\perp$. Now let $B$ be the subset of pNA of all games $q$ for which

$$(6.23) \qquad \psi(u \cdot q) = \psi_{(a,g)}(u \cdot q).$$

By (6.22) $\nu^k-\mu^k \in B$. Observe that $u \cdot \mu^k-\alpha^k \cdot u$ is in pNA and hence
$\psi(u \cdot \mu^k-u)(S) = \int_{\alpha}^{1} g(t) \cdot \partial(\mu^k)^*(t,S) \cdot dt$ and $\psi(\alpha^k u) = \alpha^k \cdot a \cdot \mu$ . Therefore
it is easily verified that $\mu^k \in B$. But $B$ is obviously a linear subspace
of pNA and therefore as it contains $\mu^k$ and $\nu^k-\mu^k$ it contains $\nu^k$ for
any probability measure in NA and hence any polynom in $NA^\perp$ measures. As
both $\psi$ and $\psi_{(a,g)}$ are continuous and $\|u \cdot q\| \leq \|u\| \cdot \|q\|$ it follows

that B is closed, thus B = pNA. Now as both $\psi$ and $\psi_{(a,g)}$ are linear and continuous we deduce that they coincide on u*pNA, which completes the proof of theorem 6.1.

Q.E.D.

## 7. Further Results and Remarks.

We are able to characterize the set of all continuous semi values on many other important spaces, like bv'NA and bv'NA*pNA. As the proof uses similar methods to those presented in the former sections we will just give a sample of results.

Notations: Let X be a linear subspace (not necessarily closed) of the banach space bv' (the space of all functions $f: [0,1] \to \mathbb{R}$ with f(0) = 0 such that f is of bounded variation continuous at zero and 1, endowed with the total variation norm). We denote by W(X) the subset of the dual $\overline{X}^*$ (of the closure $\overline{X}$ of X) of all elements $x^*$ satisfying: (1) For each monotonic nondecreasing f in X, $x^*(f) \geq 0$; (2) If X contains the function h defined by h(x) = x, then $x^*(h) = 1$. The subspace of all absolutely continuous elements in bv' is denoted ac'. For each 0 < x < 1 define $f_x: [0,1] \to \mathbb{R}$ by $f_x(y) = 0$ iff y < x and $f_x(y) = 1$ iff $y \geq x$ and $\overline{f}_x: [0,1] \to \mathbb{R}$ by $\overline{f}_x(y) = 0$ iff $y \leq x$ and $\overline{f}_x(y) = 1$ iff y > x. The subspace of bv' generated by the functions $f_x$ ($\overline{f}_x$) is denoted by rj' ($\ell j'$), and that generated by all jump functions (i.e., by rj' and $\ell j'$) is denoted by j'. If $X \subset bv'$ we denote by XNA the linear symmetric space generated by game of the form $f \circ \mu$, $f \in X$ and $\mu$ is a probability measure in NA.

Theorem 7.1: Let X be a subspace of bv'. There is a 1-1 linear isometry from W(X) onto the continuous semi values on XNA; for each $x^* \in W(X)$ the semi value $\psi_{x^*}$ on XNA is given by

$$\psi_{x^*}(f \circ \mu) = x^*(f) \cdot \mu$$

<u>Remarks</u>: (a) W(ac') = W and therefore theorem 7.1 can be considered a generalization of theorem 5.6. (ac'NA is dense in pNA).

(b) W(rj') is identified with all bounded functions a: (0,1) → $R^1$; for 0 < x < 1 $x^*(a)(f_x) = a(x)$ and $\|x^*(a)\| = \sup_{0<x<1} a(x)$. Each of the continuous semi values on rj'NA can be extended to a semi value on its closure: However, there are discontinuous semi values on rj'NA; they can be obtained by omitting the boundness condition on a.

(c) W(j') is identified with all pairs of bounded functions a,b: (0,1) → $R^1$ where for 0 < x < 1 $x^*(a,b)(fx) = a(x)$ and $x^*(a,b)(fx) = b(x)$. We have $\|x^*(a,b)\| = \sup_{0<x<1} a(x), b(x)$.

<u>Notations</u>: If $Q_1$ and $Q_2$ are linear symmetric subspaces of BV we denote by $Q_1 \circledast Q_2$ the linear symmetric space generated by games of the form $v_1 \cdot v_2$ where $v_i \in Q_i$ (i = 1,2), and the space $Q_1 * Q_2$ is defined as the linear symmetric space generated by $Q_1 \circledast Q_2$, $Q_1$ and $Q_2$.

<u>Theorem 7.2</u>: For each pair (a,g), a: (0,1) → $R^1$ and g ∈ W = W(ac') there is a semi value $\psi_{(a,g)}$ on rj'NA*pNA given by:

(7.3)             $\psi_{(a,g)}(v) = \psi_g v$ whenever v ∈ pNA

(7.4)   $\psi_{(a,g)}((f_x \circ \mu) \cdot v)(S) = a(x) \cdot v^*(x) \cdot \mu(s) + \int_x^1 g(t) \cdot \partial v^*(t,S) \cdot dt$

whenever v ∈ pNA, 0 < x < 1 and μ is a probability measure in NA. The semi value $\psi_{(a,g)}$ is continuous iff a is bounded. Moreover, any continuous semi value on rj'NA*pNA is of that form. $\psi_{(a,g)}$ can be extended to a semi value on $\overline{\text{rj'NA*pNA}}$ iff a is bounded and then

$\|\psi_{(a,g)}\| = \max(\sup_{0<x<1} a(x), \|g\|_{L_\infty})$.

<u>Remark</u>: Similar results hold for the spaces ℓj'NA*pNA and j'NA*pNA (In the second case the semi values are associated with triples (a,b,g)).

**Theorem 7.5:** For each pair $(a,g)$, $a: (0,1) \to R^1$ and $g \in L^1_\infty(0,1)$ there is a semi value $\psi_{(a,g)}$ on $rj'NA \circledast pNA$ given by (7.4). This semi value is continuous if and only if $a$ is bounded. Moreover, any continuous semi value if of that form.

**Remarks:** (a) The semi values on $rj'NA*pNA$ differ from those on $rj'NA \circledast pNA$ since $NA \subset rj'NA \circledast pNA$ while $NA \subset rj'NA*pNA$.

(b) The proof of theorems 7.2 and 7.5 are similar to that of theorem 6.1.

(c) The fact that $(a,0)$ is a semi value on $rj'NA \circledast pNA$ is easy to prove (see lemma 6.5 (6.8)) and actually makes use only on the property of $pNA$ of having a continuous extension to ideal sets satisfying lemma 6.2. Thus it follows that the existence of such semi values is valid for any space of the form $rj'NA \circledast Q$ where $A$ has such an extension. If $Q$ is such a space satisfying: there exist $\alpha: (0,1) \to R^1 \setminus \{0\}$ s.t. for each $v \in Q$ and $0 < x < 1$ $v^*(x) = \alpha(x) \cdot v^*(1)$ then by setting $a(x) = 1/\alpha(x)$, $\psi_{(a,0)}$ is a value on $rj'NA \circledast Q$. However, these values are discontinuous.

**General remarks:** (a) In a coming paper we intend to study asymptotic semi values and its relation to the core.

(b) Theorem 2.1 may be reformulated by means of "the multilinear extension" of a game (due to Owen [10]). The fact that such operators are indeed semi values was mentioned by Dubey [5].

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An axiomatic study of value theory without efficiency is presented. A characterization of the class of operators that is obtained by omitting from the values axioms the one about efficiency is given. This is done for the finite case as well as for some important spaces of non atomic games.

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